Applicability of the Uniformly Expanding Universe Model to Laser Produced Plasmas

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The similarity model commonly used in laser plasma calculations is shown to be identical with the uniformly expanding universe model analysed by Heckmann [1]. A solution is found to the corresponding gasdynamic equations also for the case of nonvanishing entropy production. The solution is analytic for a class of heat input rates. Heat input is shown to affect the gas motion in an exponential manner. The limitations on the applicability of this model to laser plasma compression and expansion calculations are discussed.

Heckmann [1], following earlier studies of Milne [2], analyzed the motion of the cosmic substance by postulating that all observers in the cosmological system must have the same "world view", i. e. the pressure, density, and velocity distributions viewed in different coordinate systems attached to the moving substrate must be the same. Hence, postulating

$$\begin{aligned} \varrho'(x'_{1i}, t) &= \varrho''(x''_{2i}, t), \\ p'(x'_{1i}, t) &= p''(x''_{2i}, t), \\ v_i'(x'_{1i}, t) &= v_i''(x''_{2i}, t), \end{aligned} \tag{1}$$

provided $x'_{1i} = x'_{2i}$, where the first subscripts 1 and 2 denote two different points and the apothrophies different coordinate systems $(i = 1, 2, 3)$, Heckmann showed that the velocity becomes a separable variable of space and time:

$$v_i = a_{ik}(t) \cdot x_k \tag{2}$$

and the state variable p, ϱ defined by the respective gasdynamic equations may be expressed as pure time functions. The solution Heckmann presented by taking gravity effects into account is known as the uniformly expanding universe model.

The same fundamental assumption, that is the separability of the space and time dependences in the velocity function serves as basis for similarily solutions proposed for gasdynamic problems that do not involve any characteristic length (spherical expansion following a point explosion, etc., see, for example: Zeldovich and Raiser [3]).

The uniform expansion model has been frequently used also for describing gasdynamic phenomena associated with laser produced plasmas without noting the cosmological origin of this approach (see, for example, Dawson [4], Hought and Polk [5], Kidder [6]).

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It shall be shown here that using the result of the cosmological postulate, i. e. the separability condition, an approximate analytical solution can be obtained to the set of gasdynamic equations:

$$\begin{aligned} \partial n/\partial t + \operatorname{div}(\boldsymbol{v}\, n) &= 0, \\ m\, n\, \partial \boldsymbol{v}/\mathrm{d}t + \operatorname{grad} p &= 0, \\ \tfrac{3}{2} n\, k'\, \mathrm{d}T/\mathrm{d}t + n\, k\, T \operatorname{div} \boldsymbol{v} &= q_{\mathrm{vol}} \end{aligned} \tag{3}$$

supplemented by the equation of state

$$p = n\, k\, T \tag{4}$$

also for non-vanishing energy input. The solution becomes exact for a class of energy input rates. Furthermore, the limitations on the applicability of this model to realistic compression and expansion calculations shall be indicated.

Considering a one-dimensional spherically symmetric gas motion in a Langrangean coordinate system the separability condition implies

$$\begin{aligned} r &= r_0\, f(t), \\ \dot r &= \mathrm{d}r/\mathrm{d}t &&= r_0\, \dot f &&= r\,(\dot f/f), \\ \ddot r &= \mathrm{d}^2 r/\mathrm{d}t^2 &&= r_0\, \ddot f &&= r\,(\ddot f/f) \end{aligned} \tag{5}$$

where dot denotes differentiation with respect to time. Note that the second of these equations is equivalent with Heckmann's result [Eq. (2)]. The initial distributions are assumed to be given:

$$\begin{aligned} n_0 &= n(t=0) &&= n_0(r_0), \\ T_0 &= T(t=0) &&= T_0(r_0), \\ v_0 &= r(t=0) &&= v_0(r_0) \end{aligned} \tag{6}$$

where $\qquad r_0 \equiv r(t=0)$.

In view of Eqs. (5) the mass continuity equation reduces to

$$\dot n/n = 3\, \dot r/r = 0 \tag{7 a}$$

which can be integrated at once yielding

$$n/n_0 = (r_0/r)^3 , \qquad (7\,b)$$

a result that was obtained already by Heckmann [1].

Eliminating $\operatorname{div} \boldsymbol{v}$ between the first and the third equations of (1) and taking Eq. (7 a) into account, the energy conservation equation may be written in the following form:

$$\dot{T}/T + 2\,\dot{r}/r = (2\,m/3\,k)\,\dot{s} \qquad (8\,a)$$

where $\dot{s} = \mathrm{d}s/\mathrm{d}t = (\varrho\, T)^{-1}\, \mathrm{d}q_{\text{vol}}/\mathrm{d}t$ represents the rate of entropy change per unit mass. Only entropy increase (heat addition) shall be considered here. Assuming that $\dot{s}$ is an analytic function of the time, Eq. (8 a) can be integrated at once to yield

$$T/T_0 = (r_0/r)^2 \exp\left(\tfrac{2}{3}\frac{m}{k}\,\Delta s\right) = \varepsilon_s (r_0/r)^2 \qquad (8\,b)$$

where

$$\Delta s = \int_0^t \mathrm{d}s \quad \text{and} \quad \varepsilon_s \equiv \exp\left(\tfrac{2}{3}\frac{m}{k}\,\Delta s\right).$$

ε_s is, in general, a function of t and r_0 that depends upon the way the energy is supplied to the system. It must, however, fullfill the conditoins

$\varepsilon_s = 1$ at $t=0$ for any energy input process; and
$\varepsilon_s = 1$ for $t \geqq 0$ in the case of an isentropic process.

In this second case Eq. (8 b) reduces to

$$T/T_0 = (r_0/r)^2 \qquad (8\,c)$$

Let us consider the momentum equation written in spherical coordinates as

$$m\, n_0 \left(\frac{r_0}{r}\right)^3 \ddot{r} + \frac{\partial}{\partial r}\left[n_0\, k\, T_0\, \varepsilon_s \left(\frac{r_0}{r}\right)^5\right] = 0 . \qquad (9)$$

Since ε_s is a time function, this equation cannot be integrated analytically except for some special cases. However, for any prescribed heat input function approximate solutions may be obtained by step-by-step integration, iterative methods, or other approximations. In such a procedure ε_s may be considered as a given quantity for the particular time interval considered.

First let us assume that

$$\varepsilon_s = (r_0/r)^\alpha, \ \text{i.e.}\ p/\varrho^n = \text{cost} \qquad (10)$$

where the polytropic exponent $n = (\alpha+5)/3$. Considering an expanding gas mass the cases $\alpha<0$, $\alpha=0$, and $\alpha>0$ correspond to flow with heat addition, adiabatic motion, and flow with heat removal, respectively. The case $\alpha = -2$ represents isothermal motion.

Substituting Eq. (10) into Eq. (9) the following expression is obtained for $f(t)$:

$$-\ddot{f}\, f^{\alpha+3} = \frac{1}{\varrho_0\, r_0}\,\frac{\partial p_0}{\mathrm{d}r_0} = \text{const} \equiv a_0 \gtreqless 0 , \qquad (11)$$

which after integrating reduces to

$$[f^{-(\alpha+2)} + \tfrac{1}{2}(\alpha+2)\,(\dot{f}_0/a_0)^2 - 1]^{-1/2}\,\mathrm{d}f = \left(\frac{2\,a_0}{\alpha+2}\right)^{\frac{1}{2}} \mathrm{d}t . \qquad (12)$$

since $t(t)$ is a pure time function, $f_0 = f(t=0) = 1$, and $\dot{f}_0 = \text{const}$, i.e.

$$\dot{r}_0/r_0 = \text{const} \equiv b_0 \gtreqless 0 . \qquad (13)$$

Equations (11) and (13) represent constraints on the admissible initial parameter distributions.

Equation (12) can be integrated once more analytically for certain α values. Assuming, for example, zero initial velocity and negative initial pressure gradient we obtain for the adiabatic case ($\alpha=0$) and for a representative case with heat addition ($\alpha = -4$)

$$f = [1 + (v_0\, t/r_0)^2]^{1/2} \quad \text{and}$$
$$f = \tfrac{1}{2}[\exp(v_0\, t/r_0) + \exp\{-v_0\, t/r_0\}] \qquad (14)$$

respectively. In these expressions

$$v_0^2 \equiv (r_0/\varrho_0)\,|\,\partial p_0/\partial r_0\,| .$$

Note the exponential expansion rate caused by heat addition as compared with the constant asymptotic value inherent in adiabatic motion.

Returning now to the general expression (9), let us assume that for the time interval considered the entropy function ε_s can be evaluated by some approximate means and be considered here as a given parameter. Equation (9) can also be integrated twice analytically in this case to yield

$$f = \left[\left(1 + \frac{\dot{r}_0}{r_0}\,t\right)^2 - \frac{\varepsilon_s}{\varrho_0\, r_0}\,\frac{\partial p_0}{\partial r_0}\,t^2\right]^{\frac{1}{2}} \qquad (15)$$

where the initial value distributions are subject to the same constraints as those given by Eqs. (11) and (13). Eq. (15) and its time derivative may thus be rewritten in the following form:

$$r/r_0 = [\,(1+b_0\, t)^2 - \varepsilon_s\, a_0\, t^2]^{1/2}, \qquad (15\,a)$$

$$\frac{\dot{r}}{r_0} = \frac{b_0 + (b_0^2 - \varepsilon_s\, a_0)\,t}{[\,(1+b_0\, t)^2 - \varepsilon_s\, a_0\, t^2]^{1/2}} . \qquad (15\,b)$$

Heat input affects the expansion rate in this case by means of the exponential multiplier ε_s present in the above expressions.

As can be seen from Eq. (13), only the linear initial velocity variation is admitted. Equation (11) yields an analytic expression for the admissible initial pressure variation, provided ϱ_0 can be expressed as $\varrho = \varrho_0(p_0)$. Assuming a polytropic dependence of the form $p_0 = \text{const.}\ \varrho_0^n$, the integral of the right side of Eq. (11) is obtained at once:

$$p_0(r_0) = p_0(0)\left[1 + \frac{n-1}{2}\, a_0 (r_0/c_{00})^2\right]^{\frac{n}{n-1}};$$

$$c_{00} = c_0(0) \equiv (p_0/\varrho_0)_{r_0=0}\,. \qquad (16)$$

Note the highly specific nature of this initial pressure distribution. Consider, for example, the spherical compression of a gaseous mass initially at rest[6]. Assuming a monatomic gas and isentropic relations ($n = \gamma = 5/3$), the required initial pressure must increase outward roughly as the fifth power of the raduis (strong pressure gradient in a gas at rest). The practical applicability of such a model is clearly limited.

In general, the motion defined by Eqs. (7 b), (8 b), and (15) may be pure compression, pure expansion, or a combination of expansion and compression phases, depending upon the initial parameter values and their gradients. A pure expansion is characterized by positive radial velocities at all times. We shall denote the asymptotic expansion velocity by u_∞: $u_\infty \equiv \dot r(t \to \infty)$. In the case of a pure compression the velocity remains negative for all times. The compression may terminate in a total mass collapse (the gas volume shrinks to zero). The time necessary for a total collapse shall be denoted by τ_0. Expansion followed by compression is characterized by a maximum to initial radius ration $r_{\max}/r_0$, the associated velocity reversal time τ_r and possibly by a following total collapse with a collapse time τ_0. Compression followed by expansion is characterized by the existance of a minimum radius $r_{\min}$, the associated velocity reversal time τ_r, and possibly by a subsequent asymptotic expansion state with an asymptotic expansion velocity u_∞. Defining a characteristic velocity v_0 by the relation

$$v_0^2 \equiv \varepsilon_s (r_0/\varrho_0) \,|\nabla p_0|\, = \varepsilon_s \,|a_0|\, r_0^2$$

the conditions leading to the different flow types may be summarized as follows (the resulting asymptotic velocities, characteristic times and/or characteristic dimensions are also given):

1. Pure expansion

1a) $\dot r_0 > 0,\ \nabla p_0 > 0,\ \dot r_0 \geqq v_0;\ u_\infty = (\dot r_0^2 - v_0^2)^{1/2};$

1b) $\dot r_0 > 0,\ \nabla p_0 = 0;\qquad u_\infty = \dot r_0;$

1c) $\dot r_0 > 0,\ \nabla p_0 < 0;\qquad u_\infty = (\dot r_0^2 + v_0^2)^{1/2};$

1d) $\dot r_0 = 0,\ \nabla p_0 < 0;\qquad u_\infty = v_0\,.$

2. Compression followed by expansion

2a) $\dot r_0 < 0,\ \nabla p_0 < 0;\ \tau_r = r_0\,|\dot r_0|/(\dot r_0^2 + v_0^2);$
$\qquad r_{\min} = r_0 v_0/(\dot r_0^2 + v_0^2)^{1/2};$
$\qquad u_\infty = (\dot r_0^2 + v_0^2)^{1/2};$

2b) $\dot r_0 < 0,\ \nabla p_0 = 0;\ \tau_r = \tau_0 = r_0/|\dot r_0|;$
$\qquad r_{\min} = 0;$
$\qquad u_\infty = |\dot r_0|.$

3. Pure compression

3a) $\dot r_0 = 0,\ \nabla p_0 > 0;\qquad \tau_0 = r_0/v_0;$

3b) $\dot r_0 < 0,\ \nabla p_0 > 0,\ |\dot r_0| < v_0;\ \tau_0 = r_0/(v_0 + |\dot r_0|).$

4. Expansion followed by compression

4a) $\dot r_0 > 0,\quad \nabla p_0 > 0,\ \dot r_0 < v_0;\ \tau_\varrho$
$\qquad \tau_r = r_0 \dot r_0/(v_0^2 - \dot r_0^2);$
$\qquad r_{\max} = r_0 v_0/(v_0^2 - \dot r_0^2)^{1/2};$
$\qquad \tau_0 = r_0/(v_0 - \dot r_0).$

It will be noted that all motions considered which terminate in expansion are characterized by the existence of a time-independent asymptotic expansion velocity. Since different initial conditions may result in the same asymptotic velocity, the asymptotic state depends only weakly on the initial conditions.

We may thus conclude that the cosmological model may yield useful results when the asymptotic state of expanding laser produced plasmas is considered. This asymptotic state is characterized by uniform expansion, the expansion rate coefficient being given by the initial plasma parameters. Heat addition affects the expansion rate coefficient in an exponential manner.

Care should be exercised in applying this model to plasma compression by laser or both the compression and expansion phases of laser heated plasma combined[7]. If compression alone is considered[6], no asymptotic state exists and only a well defined initial state parameter distribution leads to the singular end state characterizing the moment of collapse. The initial distribution necessary for initiating such a process is not easy to realize. If the compression and expansion phases are considered together, an exact set of initial conditions is to be prescribed for the beginning of the compression phase: the solution is then uniquely defined up to the asymptotic expansion state. Attempts to specific quantities at the end of the compression phase or to introduce additional constraints (the assumption of isothermal expansion in [7]) are in this case erroneous.

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